Journal of Engineering Physics and Thermophysics, Vol. 78, No. 5, 2005

SIMULATION OF THE MOTION OF AN INDIVIDUAL ERYTHROCYTE IN A NARROW CAPILLARY

Z. P. Shul'man, A. A. Makhanek, and S. A. Gubarev

UDC 532.5

The motion of an individual deformable erythrocyte in a capillary whose diameter is smaller than the particle diameter has been considered. The problem formulation is based on the Lighthill–Fitzgerald model. The erythrocyte drag at a given pressure difference acting on its frontal surface has been determined. The dependences of the relative drag on the dimensionless parameters of the model as well as on the minimum thickness of the lubricating layer and the rate of motion of the erythrocyte on the pressure at opposite edges of the cell have been obtained.

Simulation of the blood flow in the microcirculatory system is one of the key problems arising in quantitative descriptions of the convective heat transfer and the thermoregulation and oxygenation of living biological tissues. As compared to the hemodynamic problems for arteries, this problem is more complicated, since the blood in tiny vessels can no longer be considered as a continuous medium. The main (greater in number) form elements of blood — erythrocytes — that comparable in diameter to capillaries, and they are sometimes larger than capillaries. Motion of red blood cells under such conditions turns out to be possible only due to the high deformability of their membranes and the formation between the surface of these cells and the capillary wall of a thin plasma layer acting as a lubricant.

The biomechanical properties of blood particles and vascular walls, as well as the blood flow in tiny vessels, have been the subject of many papers, including [1–6]. In [2], it was shown that in the gap between two erythrocytes moving through a capillary (the so-called "bolus") plasma should circulate in the direction of the particle motion on the axis and in the reverse direction by the wall. In [4], the deformation of erythrocytes in capillaries depending on the vessel diameter and the rate of motion of particles was investigated experimentally. The ratios were obtained and the velocity profile of plasma in the bolus was calculated. In [5], the motion of erythrocytes through capillaries with diameters close to critical ones was investigated. The size of the latter depends on the geometric and mechanical properties of the red blood cells: they deform at a constant volume (the liquid inside erythrocytes is incompressible) and an almost constant area of the surface (the erythrocytic membrane is poorly stretchable). In [6], the motion of a suspension of elastic incompressible spheres through a capillary was considered. The development of this model was motivated by the interest in investigating the motion of leukocytes (white blood cells) through the microcirculatory system.

One of the earliest physicomathematical models of blood flow in narrow capillaries is the Lighthill model [7] constructed in the approximations of lubrication theory. The local elastic properties of the vessel wall and the erythrocyte in the first approximation in [7] were assumed to be proportional to the excess pressure. The equation of motion was reduced to the Reynolds equation for the lubricating layer. The Lighthill model was further developed by Fitzgerald [8]. The latter, assuming that the erythrocyte has a finite length, considered the elastic properties of this cell in more detail and obtained a solution of the equation of motion for the case of axial symmetry of the erythrocyte, as well as in the absence of such a symmetry and in the presence of filtration liquid flows through the penetrable capillary walls.

The simulation of the erythrocyte motion in a narrow capillary attempted in [7, 8] and later in [9, 10] was not carried to the calculation of the hemodynamic erythrocyte and plasma drag created by the narrow capillary walls.

A. V. Luikov Heat and Mass Transfer Institute, National Academy of Sciences of Belarus, 15 P. Brovka Str., Minsk, 220072, Belarus. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 78, No. 5, pp. 5–9, September–October, 2005. Original article submitted November 1, 2004.

The analysis in [7, 8] was performed with the use of dimensionless complexes characterizing the erythrocyte velocity and the relative value of the plasma flow. These and some other parameters were considered to be independent, and the final result of the calculations according to the model proposed by Lighthill was the relative value of the pressure at the distal end of the erythrocyte. The aim of the present work is to simulate the hemodynamic drag of an individual erythrocyte at given values of the pressure drop at its ends, as well as at other geometric and rheophysical parameters.

Calculation of the Motion of an Individual Erythrocyte in a Capillary. Our numerical analysis of the motion of an individual erythrocyte in a capillary is based on the Lighthill model in Fitzgerald's interpretation [8]. We assume the particle (erythrocyte) to be finite in integrating the continuity equation. According to Lighthill's hypothesis, the gap width between the erythrocyte membrane surface and the capillary wall is described by the expression

$$h(x) = r_0 - R(x) + (\alpha + \beta)\,[p(x) - p_0]\,. \tag{1}$$

For the skin capillaries, we assume $\alpha = 0$, i.e., the capillary walls do not deform under the action of pressure.

The erythrocyte profile is often modeled in the form of a parabola as the most suitable approximation of the parachute-like (or bullet-like) shape characteristic of its motion in a capillary:

$$R(x) = r_0 \sqrt{1 - x^2\kappa/r_0} - \beta\,(p - p_0)\,. \tag{2}$$

According to Fitzgerald, the pressure p_0 can be determined from the relation

$$\frac{\beta}{r_0}\left(p_0 - \frac{p(-g) + p(g)}{2}\right) = R_c - 1\,, \tag{3}$$

where

$$R_{\tilde{n}} = \frac{2(1+n)}{15E} \sin \frac{15aE}{4(1+n)\,r_0} \quad (n \sim 2\text{–}3\,,\ a \approx 7.5\ \mu\text{m})\,; \tag{4}$$

$$E = \beta\,\frac{p(-g) - p(g)}{r_0}\,.$$

If E is small enough (at r_0 = 4 μm $E < 0.3$), then $R_c \approx a/(2r_0)$, i.e., this parameter is approximately equal to the ratio of the rear part of the erythrocyte membrane to the capillary diameter. When the erythrocyte size in the stress-free state is larger than the values of the capillary clearance, the parameter $R_c > 1$ and, vice versa, if the size of the stress-free erythrocyte is smaller than the capillary clearance, then $R_c < 1$.

The parameter g entering into expression (3) represents the coordinate of the distal end of the erythrocyte and is determined from the condition of vanishing of the particle radius here:

$$g(x) = \left(\frac{r_0}{\kappa}\right)^{1/2} \left\{1 - E^2\left(\frac{p(-g) - p_0}{\Delta p}\right)^2\right\}^{1/2}, \tag{5}$$

where $\Delta p = p(-g) - p(g)$.

According to the methodology of lubrication theory, Lighthill (and Fitzgerald) obtained the Reynolds equation (analog of the equation of motion for the gap-width-averaged plasma velocity) in the form

$$\frac{dp}{dx}\,\frac{2Rh + h^2}{16\mu}\left[2R^2 + 2Rh + h^2 - \frac{2Rh + h^2}{\ln(1 + h/R)}\right] + U\left[\frac{(R+h)^2}{2} - \frac{2Rh + h^2}{2\ln(1 + h/R)}\right] = r_0 Q\,. \tag{6}$$

The balance of the forces acting at opposite ends of the particle is described by the following expression:

$$\pi r_0^2 \Delta p = 2\pi \int_{-g}^{g} \left\{ -\frac{1}{4} \frac{dp}{dx} \left[2R^2 - \frac{2Rh + h^2}{\ln(1 + h/R)} \right] + \frac{\mu U}{\ln(1 + h/R)} \right\} dx \, . \tag{7}$$

Unlike the above-mentioned works, we dedimensionalize the above equations in a radically different way without introducing complexes depending simultaneously on two unknown parameters — the erythrocyte velocity U and the plasma flow Q:

$$P = \frac{p - p_0}{\Delta p}, \quad \rho = R/r_0 , \quad \chi = h/r_0 = 1 - \rho , \quad u = \frac{2L_c\mu}{\Delta p r_0^2} U , \quad q = \frac{2L_c\mu}{\Delta p r_0^3} Q , \quad G = \frac{g}{L_c/2} ,$$

$$X = \frac{x}{L_c/2} , \quad f_1 = 2\rho^2 + \frac{1 - \rho^2}{\ln \rho} , \quad f_2 = \frac{1 - \rho^4}{4} + \frac{(1 - \rho^2)^2}{4 \ln \rho} , \quad f_3 = \frac{1}{2} + \frac{1 - \rho^2}{4 \ln \rho} .$$

As a result, the Reynolds equation and the mass equation take the form

$$\frac{dP}{dX} = \frac{q}{f_2} - \frac{uf_3}{f_2} , \tag{8}$$

$$\frac{1}{2} \int_{G}^{-G} \left\{ \left[q - uf_3 \right] \frac{f_1}{f_2} + \frac{u}{\ln \rho} \right\} dX = 1 , \tag{9}$$

$$\rho(X) = \sqrt{1 - X^2} - EP , \quad G = \sqrt{1 - (EP_{-G})^2} ,$$

$$P_{-G} = \frac{p(-g) - p_0}{\Delta p} = \frac{1}{2} - \frac{R_c - 1}{E} , \quad p_0 = p + \Delta p \frac{R_c - 1}{E} .$$

The condition of compatibility of initial data is the inequality $E < 1/\left|P_{-G}\right|$. For this inequality to be fulfilled at small values of E, it is essential that R_c be smaller than 2. If $R_c < 1$, then the erythrocyte will pass through the capillary at any pressure and a positive drop of pressure at the capillary ends.

To find the unknown parameters u and q, we take into account that they are independent of X and, consequently, can be removed from the integral sign in (9). As a result, we find one of the equations needed to determine these parameters. The second equation is obtained after integrating expression (8) with respect to the gap width between the erythrocyte membrane and the capillary wall. In the final analysis, we have the following system of algebraic equations:

$$A_{11}u + A_{12}q = 1 , \quad A_{21}u + A_{22}q = 1/2 \tag{10}$$

with coefficients

$$A_{11} = \frac{1}{2} \int_{-G}^{G} \left(\frac{f_1 f_3}{f_2} - \frac{1}{\ln \rho} \right) dX , \quad A_{12} = -\frac{1}{2} \int_{-G}^{G} \frac{f_1}{f_2} dX , \quad A_{21} = \frac{1}{2} \int_{-G}^{G} \frac{f_3}{f_2} dX , \quad A_{22} = -\frac{1}{2} \int_{-G}^{G} \frac{dX}{f_2} . \tag{11}$$

In the general case, this system is nonlinear, since the relative radius of the erythrocyte depends on the pressure, whose profile is connected with the sought parameters u and q. We have realized a computer program for determining the erythrocyte velocity and the plasma flow at a given pressure drop on the erythrocyte. The algorithm is based on the solution of the system of equations (10) at some given (as the initial approximation) pressure profile. The new pressure profile is determined from the solution of the differential equation (8) with the found approximate values of

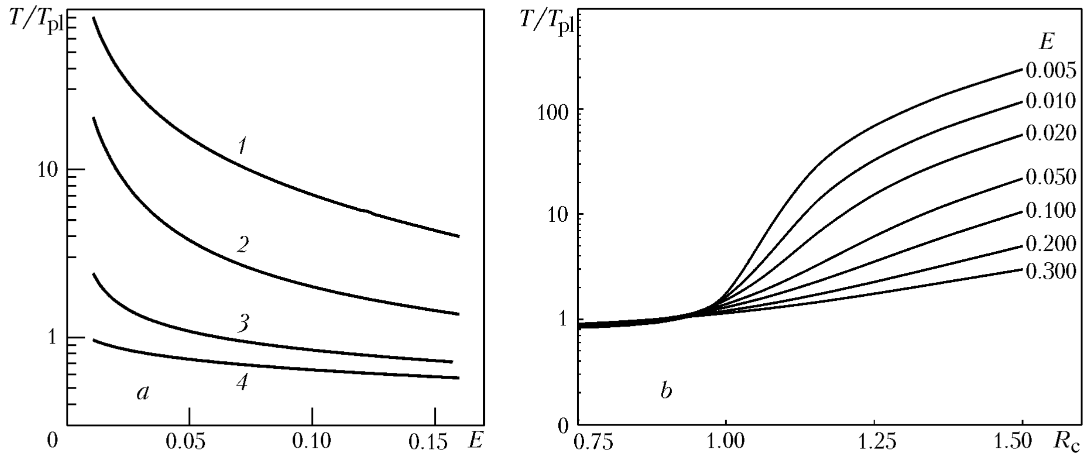

Fig. 1. Dependence of the relative drag on the parameters E (a) [1), capillary radius of 2.5; 2) 3.75; 3) 5; 4) 3.75 μm] and R_c (b).

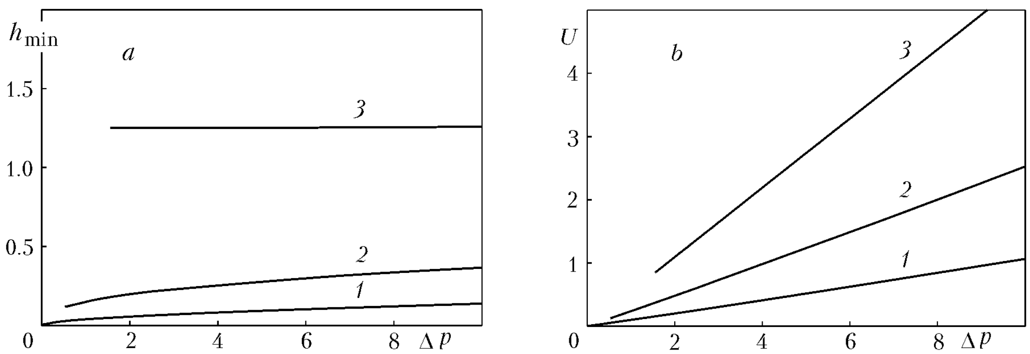

Fig. 2. Influence of pressure drops at the erythrocyte ends on the minimum thickness of the plasma layer between the erythrocyte membrane and the capillary wall (a), as well as on the erythrocyte velocity (b): 1) capillary radius of 2.5; 2) 3.75; 3) 5 μm. $h_{\min}$, μm; Δp, Torr; U, mm/sec.

the parameters u and q. These parameters and the pressure distribution are redetermined in the iteration process until the known (given) value of the pressure at the distal end of the capillary is obtained.

Results and Discussion. Figure 1 shows the dependences on the dimensionless complexes E and R_c of the relative drag as a single erythrocyte is moving in the distal part of the capillary compared to the drag of a portion of plasma of the same extent as the erythrocyte. A sharp increase in this relative drag at small E and small capillary radii or at large values of the complex R_c can be stated. In the case of fairly large sizes of the capillary, the ratio T/T_{pl} can turn out to be less than unity, which apparently indicates that such flow conditions are inconsistent with the approximation of the thin layer between the erythrocyte membrane and the capillary wall, in which the Lighthill–Fitzgerald model was constructed.

The minimum thickness of the plasma layer between the erythrocyte membrane and the capillary wall, as well as the erythrocyte velocity, are determined by the value of the pressure drop at the erythrocyte ends. The dependence of these parameters on Δp is shown in Fig. 2. Noteworthily (see Fig. 2a), the dependence of the plasma-layer thickness on Δp is revealed only at an insignificant pressure difference. For the physiological conditions and at a given value of the medium pressure in the region where the erythrocyte is located, the plasma-layer thickness turns out to be weakly dependent on Δp. The numerical results presented in Figs. 1 and 2 were obtained at a value of this pressure equal to 10 Torr. For the simulation conditions the erythrocyte velocity is proportional to the pressure drop on the cell.

NOTATION

a, arc length of the rear part of the erythrocyte, m; *E*, dimensionless parameter characterizing the pressure drop on the erythrocyte; *G*, dimensionless coordinate of the distal end of the erythrocyte; *g*, coordinate of the distal end of the erythrocyte, m; $h(x)$, gap width between the erythrocyte membrane surface and the capillary wall, m; L_c, erythrocyte length, m; *n*, dimensionless coefficient equal to the ratio of the radii of curvature of the outer (frontal) and inner (rear) surface of the erythrocyte; *P*, dimensionless pressure; *p*, pressure, Pa; p_0, characteristic pressure (at which the erythrocyte would completely fill the capillary clearance), Pa; *Q*, value of the flow (volumetric rate of flow of plasma forced against the erythrocyte motion assigned to the perimeter of the capillary cross section), m^2/sec; *q*, dimensionless flow value; $R(x)$, stress-free erythrocyte profile, m; R_c, dimensionless parameter characterizing mechanical properties of the erythrocyte; r_0, capillary radius; *T*, flow drag, Pa·sec/m^3; *U*, erythrocyte velocity, m/sec; *u*, dimensionless erythrocyte velocity; *X*, dimensionless length along the vessel axis *x*, longitudinal (along the vessel axis) coordinate, m; α, compliance coefficient of the capillary wall, m/Pa; β, compliance coefficient of the erythrocyte membrane, m/Pa; Δp, pressure drop, Pa; κ, dimensionless curvature of the meridional section of the erythrocyte at the point with its maximum diameter; μ, plasma viscosity, Pa·sec; ρ, dimensionless particle profile; χ, dimensionless lubricating layer thickness of plasma. Subscripts: pl, plasma; c, cell (erythrocyte); min, minimum value of the parameter.

REFERENCES

1. Y. C. Fung, Theoretical considerations of the elasticity of red cells and small blood vessels, *Fed. Proc.*, **25**, 1761–1772 (1966).
2. A. C. Burton, Role of geometry, of size and shape, in the microcirculation, *Fed. Proc.*, **25**, 1753–1760 (1966).
3. R. S. Frank and R. M. Hochmuth, The influence of red cell mechanical properties on flow through single capillary-sized pores, *J. Biomech. Eng.*, **110**, 155–160 (1988).
4. Y. C. Fung, Blood flow in the capillary bed, *J. Biomech.*, **2**, 353–372 (1969).
5. D. Halpern and T. W. Secomb, The squeezing of red blood cells through capillaries with near-nominal diameters, *J. Fluid Mech.*, **203**, 381–400 (1989).
6. H. Tözeren and R. Skalak, The steady flow of closely fitting incompressible elastic spheres in a tube, *J. Fluid Mech.*, **87**, Pt. 1, 1–16 (1978).
7. M. J. Lighthill, Pressure-forcing of tightly fitting pellets along fluid-filled elastic tubes, *J. Fluid Mech.* [Russian translation], No. 6, 42–70 (1970).
8. J. M. Fitzgerald, Mechanics of red-cell motion through very narrow capillaries, *J. Fluid Mech.* [Russian translation], No. 4, 90–118 (1971).
9. R. Ducharme, P. Kapadia, and J. Dowden, A mathematical model of the flow of blood cells in fine capillaries, *J. Biomech.*, **24**, No. 5, 299–306 (1991).
10. V. A. Kalion and Yu. I. Shmakov, Mathematical model of motion of erythrocytes through capillaries and its application to determination of elastic properties of the shell, *Dokl. Akad. Nauk Ukr.SSR*, *Ser. A*, *Fiz.-Mat. Tekh. Nauki*, No. 4, 49–52 (1990).